Chapter 7

Random Variables

Topics to Cover

- Motivation for random variables
- Definition as a mapping
- Domain, range, and codomain
- Direct and inverse images
- Discrete random variables
- Probability mass function (PMF)

Learning Objectives

- Understand why random variables are introduced.
- Define a random variable as a function.
- Interpret mappings from outcomes to numbers.
- Compute probabilities using inverse images.
- Construct a probability mass function.

7.1 Motivation: assigning numerical values

Consider a simple experiment: tossing a coin.

$$\Omega = \{H, T\}$$

Instead of working directly with outcomes, we assign a number to each outcome:

$$H \mapsto 1, \quad T \mapsto 0$$

This assignment allows us to study the experiment using numerical values.

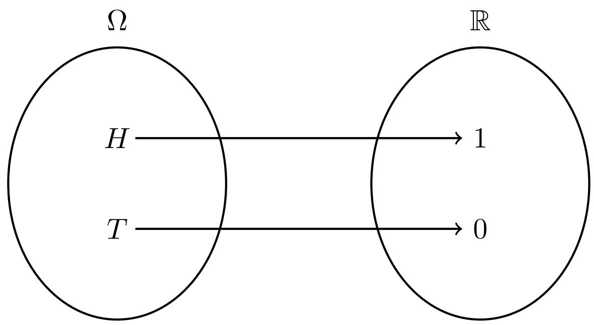

Figure 7.1: Mapping outcomes to numerical values

7.2 Definition of a random variable

Definition. A **random variable** is a function:

$$X : \Omega \to \mathbb{R}$$

Each outcome $\omega \in \Omega$ is mapped to a real number:

$$\omega \mapsto X(\omega)$$

7.3 Random variables as functions between sets

We view a random variable as a mapping between two sets.

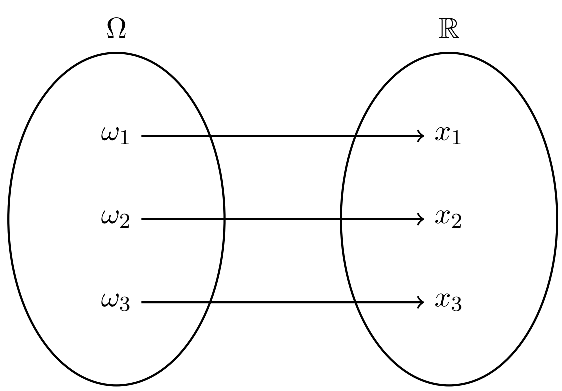

Figure 7.2: Random variable as a mapping from Ω to $\mathbb{R}$

7.4 Example: rolling a die

Consider the experiment of rolling a die:

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

Define a random variable:

$$X(\omega) = \omega$$

This assigns to each outcome its numerical value.

Another example. Define a different random variable:

$$X_2(\omega) = \begin{cases} 1 & \text{if } \omega \in \{4, 5, 6\} \\ 0 & \text{if } \omega \in \{1, 2, 3\} \end{cases}$$

This random variable indicates whether the outcome is large.

7.5 Visualization of a random variable on a die

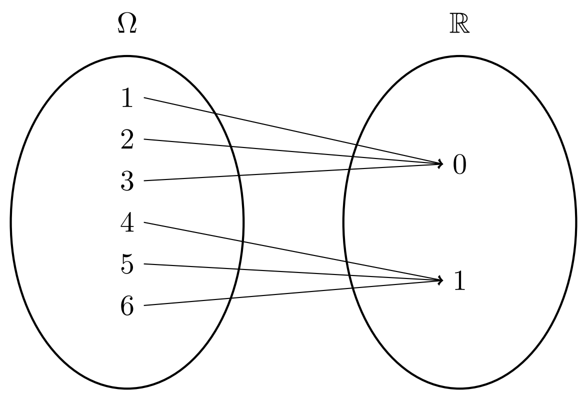

Figure 7.3: Indicator random variable: mapping outcomes to $\{0, 1\}$

7.6 Domain, codomain, and range

For a random variable:

$$X : \Omega \to \mathbb{R}$$

- Domain: the set Ω
- Codomain: the set $\mathbb{R}$
- Range: the set of values actually taken by X

Example. For:

$$X_2(\omega) = \begin{cases} 1 & \omega \in \{4, 5, 6\} \\ 0 & \text{otherwise} \end{cases}$$

the range is:

$$\{0, 1\}$$

7.7 Different random variables on the same experiment

Different random variables can be defined on the same sample space.

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

Examples:

$$X(\omega) = \omega$$

$$X_2(\omega) = \begin{cases} 1 & \omega \in \{4, 5, 6\} \\ 0 & \text{otherwise} \end{cases}$$

These represent different ways of extracting information from the same experiment.

7.8 Direct and inverse images

Let $A \subseteq \Omega$ and $B \subseteq \mathbb{R}$.

7.8.1 Direct image

The direct image of a subset $A \subseteq \Omega$ is:

$$X(A) = \{X(\omega) : \omega \in A\}$$

This gives the set of values taken by X on A.

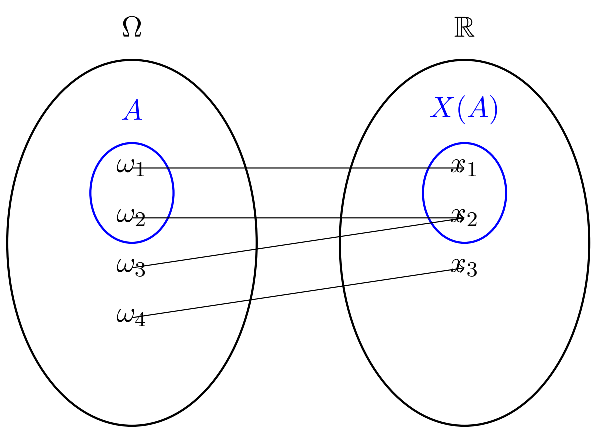

Figure 7.4: Direct image: $A = \{\omega_1, \omega_2\}$ maps to $X(A) = \{x_1, x_2\}$

7.8.2 Inverse image

The inverse image of a subset $B \subseteq \mathbb{R}$ is:

$$X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\}$$

This gives the set of outcomes that map into B.

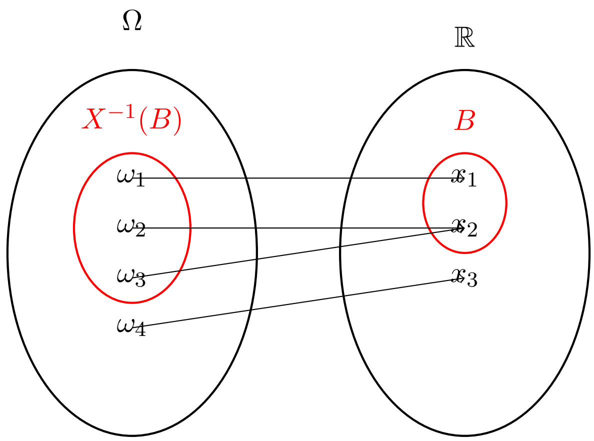

Figure 7.5: Inverse image: $B = \{x_1, x_2\}$ yields $X^{-1}(B) = \{\omega_1, \omega_2, \omega_3\}$

7.9 Probabilities using inverse images

For any set $B \subseteq \mathbb{R}$:

$$P(X \in B) = P\big(X^{-1}(B)\big)$$

7.10 Example: coin toss

$$\Omega = \{H, T\}, \quad X(H) = 1, \quad X(T) = 0$$

$$P(X = 1) = P(\{\omega : X(\omega) = 1\}) = P(\{H\})$$

$$P(X \in \{0, 1\}) = P(\{H, T\}) = 1$$

7.11 Discrete random variables

Definition. A random variable X is called **discrete** if it takes values in a finite or countable set:

$$\{x_1, x_2, x_3, \dots\}$$

7.12 Probability mass function (PMF)

Definition. The probability mass function (PMF) of X is defined as:

$$p_X(x) = P(X = x)$$

Using inverse images.

$$p_X(x) = P(\{\omega : X(\omega) = x\})$$

7.13 Example: number of heads

Consider two coin tosses:

$$\Omega = \{HH, HT, TH, TT\}$$

Define:

$$X = \text{number of heads}$$

$$X(HH) = 2, \quad X(HT) = 1, \quad X(TH) = 1, \quad X(TT) = 0$$

Thus:

$$X \in \{0, 1, 2\}$$

$$p_X(0) = P(\{TT\}) = \frac{1}{4}$$

$$p_X(1) = P(\{HT, TH\}) = \frac{1}{2}$$

$$p_X(2) = P(\{HH\}) = \frac{1}{4}$$

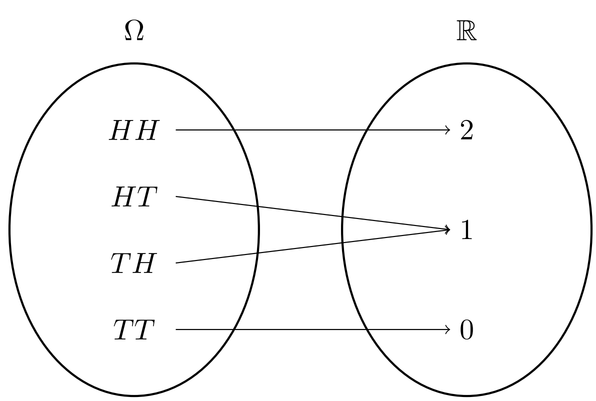

Figure 7.6: Random variable: number of heads

7.14 Properties of the PMF

- $p_X(x) \geq 0$
- $\sum_x p_X(x) = 1$

7.15 Modeling checklist

1. Define the sample space Ω
2. Define the random variable $X : \Omega \to \mathbb{R}$

3. Identify the range of X
4. Compute inverse images:
$$X^{-1}(x)$$
5. Compute:
$$p_X(x) = P(X = x)$$

KEEN 3Cs

Curiosity

How does mapping outcomes to numbers simplify probability problems?

Connections

Random variables connect sample spaces to numerical representations of uncertainty.

Creating Value

Define a random variable for an experiment and compute its PMF.

Practice

A. Remember and Understand

1. [Remember]Define a discrete random variable.
2. [Understand]Define the PMF.

B. Apply and Analyze

1. [Apply]Compute a PMF from a given experiment.
2. [Analyze]Use inverse images to compute probabilities.

C. Evaluate and Create

1. [Evaluate]Compare two different random variables on the same space.
2. [Create]Construct a PMF for a real system.