Chapter 6

Independence of Events

Topics to Cover

- Definition of independence
- Independence via conditional probability
- Product rule for independent events
- Independence and complements
- Pairwise and mutual independence

Learning Objectives

- Define independence using conditional probability and joint probability.
- Verify independence using multiple equivalent conditions.
- Apply independence properties to complements and derived events.
- Distinguish between pairwise and mutual independence.

6.1 Definition of independence

Let A and B be events.

Definition (via conditioning). Assume $P(A) > 0$. B is independent of A if:

$$P(B \mid A) = P(B)$$

Similarly, if $P(B) > 0$, A is independent of B if:

$$P(A \mid B) = P(A)$$

Interpretation. Independence means that knowledge of one event does not change the probability of the other.

6.2 Equivalent formulation

Proposition. If $P(A) > 0$ and $P(B) > 0$, then:

$$P(B \mid A) = P(B) \iff P(A \cap B) = P(A)P(B)$$

Proof. From conditional probability:

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

Thus:

$$P(B) = \frac{P(A \cap B)}{P(A)} \quad \Rightarrow \quad P(A \cap B) = P(A)P(B)$$

Conversely, if:

$$P(A \cap B) = P(A)P(B)$$

then:

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = P(B)$$

6.3 Symmetry of independence

Proposition. If $P(A) > 0$ and $P(B) > 0$, then:

$$P(B \mid A) = P(B) \iff P(A \mid B) = P(A)$$

Proof. If:

$$P(A \cap B) = P(A)P(B)$$

then:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$$

Thus independence is symmetric.

6.4 Independence and complements

Proposition. If A and B are independent, then:

- A^c and B are independent
- A and B^c are independent
- A^c and B^c are independent

Proof. First, consider A^c and B:

$$P(A^c \cap B) = P(B) - P(A \cap B)$$

Using independence:

$$= P(B) - P(A)P(B) = P(B)(1 - P(A)) = P(B)P(A^c)$$

Thus:

$$P(A^c \cap B) = P(A^c)P(B)$$

which shows independence.
The other cases follow similarly.

6.5 Conditional probabilities under independence

Proposition. If A and B are independent and $P(A) > 0$:

$$P(B \mid A) = P(B)$$

If $P(A^c) > 0$:

$$P(B \mid A^c) = P(B)$$

Proof.

$$P(B \mid A^c) = \frac{P(B) - P(A \cap B)}{P(A^c)} = \frac{P(B) - P(A)P(B)}{1 - P(A)} = P(B)$$

Thus independence extends to complements.

6.6 Basic properties

Property. If A and B are independent:

$$P(A \cap B) = P(A)P(B)$$

Property. If A and B are independent:

$$P(A \mid B) = P(A), \quad P(B \mid A) = P(B)$$

6.7 Multiple events

6.7.1 Pairwise independence

Events A, B, and C are pairwise independent if:

$$P(A \cap B) = P(A)P(B)$$

$$P(B \cap C) = P(B)P(C)$$

$$P(A \cap C) = P(A)P(C)$$

6.7.2 Mutual independence

Events A, B, and C are mutually independent if:

- they are pairwise independent, and
- $$P(A \cap B \cap C) = P(A)P(B)P(C)$$

Important. Pairwise independence does not imply mutual independence.

6.8 Extension to n events

Events $A_1, A_2, \ldots, A_n$ are independent if for every subset:

$$P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = P(A_{i_1})P(A_{i_2}) \cdots P(A_{i_k})$$

for all $k \geq 2$.

6.9 Modeling checklist

1. Identify events A, B
2. Compute:
$$P(A), \quad P(B), \quad P(A \cap B)$$
3. Check:
$$P(A \cap B) = P(A)P(B)$$
4. Alternatively verify:
$$P(A \mid B) = P(A)$$

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Curiosity

Can two events appear related but still be independent? Find examples where intuition is misleading.

Connections

Independence builds on:

- Conditional probability (Chapter 4)
- Bayes' theorem (Chapter 5)

Creating Value

Construct an example of three events that are pairwise independent but not mutually independent.

Practice

A. Remember and Understand

1. [Remember]Define independence.
2. [Understand]State the product rule for independent events.

B. Apply and Analyze

1. [Apply]Check whether given events are independent.
2. [Analyze]Verify independence for complements.

C. Evaluate and Create

1. [Evaluate]Explain the difference between pairwise and mutual independence.
2. [Create]Construct events that are pairwise but not mutually independent.