Chapter 4

Conditional Probability and the Law of Total Probability

Topics to Cover

- Conditional probability and its interpretation
- Multiplication rule for joint events
- Partition of the sample space
- Law of Total Probability
- Conditioning as information update

Learning Objectives

- Define and compute conditional probabilities.
- Derive the multiplication rule for probabilities.
- Use partitions to decompose complex probability calculations.
- Apply the Law of Total Probability in structured settings.
- Interpret conditioning as updating beliefs given new information.

4.1 Motivation: Updating with information

In earlier chapters, probabilities were assigned to events before observing additional information. In many engineering settings, information arrives sequentially.

Examples:

- A sensor reports a signal.
- A test result becomes available.

- A partial observation is revealed.

We now formalize how probabilities change when new information is known.

4.2 Conditional probability

Definition. Let A and B be events with $P(B) > 0$. The **conditional probability** of A given B is:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}.$$

Interpretation.

- We restrict attention to outcomes in B
- We renormalize probabilities within B

Key identity.

$$P(A \cap B) = P(A \mid B)\, P(B).$$

This is called the **multiplication rule**.

4.3 Examples of conditional probability

4.3.1 Card example

Draw one card from a standard deck.

Let:

$$A = \{\text{card is a heart}\}, \quad B = \{\text{card is red}\}.$$

Then:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{13/52}{26/52} = \frac{1}{2}.$$

4.3.2 System reliability

Let:

$$A = \{\text{component 1 works}\}, \quad B = \{\text{component 2 works}\}.$$

Then:

$$P(A \mid B)$$

represents the reliability of component 1 given component 2 is functioning.

4.4 Multiplication rule and sequential structure

The multiplication rule generalizes:

$$P(A \cap B \cap C) = P(A \mid B \cap C)\, P(B \mid C)\, P(C).$$

This rule allows us to compute probabilities of multi-stage processes.

Example (two-step experiment).

$$P(A \cap B) = P(A)\,P(B \mid A).$$

4.5 Partitions of the sample space

A collection of events $\{B_1, B_2, \ldots, B_n\}$ is a **partition** if:

- $B_i \cap B_j = \emptyset$ for $i \neq j$
- $\bigcup_{i=1}^{n} B_i = \Omega$

Partitions represent mutually exclusive, exhaustive scenarios.

4.6 Law of Total Probability

Let $\{B_1, B_2, \ldots, B_n\}$ be a partition with $P(B_i) > 0$. Then for any event A:

$$P(A) = \sum_{i=1}^{n} P(A \mid B_i)\,P(B_i).$$

Interpretation. We compute $P(A)$ by:

- conditioning on which scenario B_i occurs
- weighting by $P(B_i)$

4.7 Examples of the Law of Total Probability

4.7.1 Manufacturing example

A factory has two machines:

- Machine 1 produces 70% of items (event B_1)
- Machine 2 produces 30% (event B_2)

Let:

$$A = \{\text{item is defective}\}.$$

Given:

$$P(A \mid B_1) = 0.01, \quad P(A \mid B_2) = 0.03.$$

Then:

$$P(A) = 0.01(0.7) + 0.03(0.3) = 0.016.$$

4.7.2 Weather example

Let:

$$B_1 = \{\text{cloudy}\}, \quad B_2 = \{\text{clear}\}.$$

Then any probability of rain can be decomposed as:

$$P(\text{rain}) = P(\text{rain} \mid \text{cloudy})P(\text{cloudy}) + P(\text{rain} \mid \text{clear})P(\text{clear}).$$

4.8 Conditioning as filtering

Conditional probability can be viewed as filtering the sample space:

- Original space: Ω
- After observing B: new space is B

Within B, probabilities are rescaled to sum to 1.

4.9 Modeling checklist

1. Identify events A, B
2. Check that $P(B) > 0$
3. Compute $P(A \cap B)$
4. Use:
$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$
5. If multiple cases exist, define a partition
6. Apply:
$$P(A) = \sum_i P(A \mid B_i)P(B_i)$$

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Curiosity

How does additional information change decision outcomes? Explore how conditioning alters probabilities in real systems.

Connections

Conditional probability connects to:

- Bayes' theorem (next chapter)
- Independence
- Random variables and distributions

Creating Value

Design a decision rule for a system where information arrives sequentially (e.g., sensor networks, testing pipelines) and justify how probabilities are updated.

Practice

A. Remember and Understand

1. [Remember]Define conditional probability.
2. [Understand]State the multiplication rule.
3. [Understand]Define a partition of a sample space.

B. Apply and Analyze

1. [Apply]A card is red. What is the probability it is a heart?
2. [Apply]Compute $P(A)$ using the Law of Total Probability given a two-case partition.
3. [Analyze]Show that $P(A \cap B) = P(A \mid B)P(B)$.

C. Evaluate and Create

1. [Evaluate]Explain how conditioning changes a probability model.
2. [Create]Construct a partition-based model for a reliability system with two modes of operation.