Chapter 3

Counting Principles and Equally Likely Outcomes

Topics to Cover

- Product rule (rule of product) for multi-step processes
- Permutations and combinations; $\binom{n}{k}$ and factorial notation
- Equally likely outcome models and basic probabilities
- Simple counting trees and sanity checks

Learning Objectives

- Decompose multi-step experiments and apply the product rule to count outcomes.
- Compute permutations and combinations; use $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$.
- Model basic systems with equally likely outcomes and compute probabilities by counting.
- Use small trees to check that counts match the process definition.

3.1 Product rule: multi-step counting

When an experiment proceeds in steps, and the number of options at each step is fixed (or can be tracked), the **product rule** gives the total number of outcomes as the product of the number of options at each step.

Example (two-step code). A device generates a two-symbol code: the first symbol is chosen from $\{A, B, C\}$ and the second from $\{0, 1\}$. There are $3 \times 2 = 6$ outcomes:

$$\{A0, A1, B0, B1, C0, C1\}.$$

3.2 Permutations and combinations

Permutation of n distinct items. Arrangements (order matters): $n! = n \cdot (n-1) \cdots 2 \cdot 1$.

k-permutations from n distinct items. Arrangements of k out of n (order matters):

$$P(n, k) = n(n-1) \cdots (n-k+1) = \frac{n!}{(n-k)!}.$$

Combinations of n choose k (order does not matter).

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Example (ID badges). Four distinct letters $\{A, B, C, D\}$; form a two-letter badge:

- If order matters (e.g., printed left-to-right), $P(4, 2) = 4 \cdot 3 = 12$.
- If order does not matter (e.g., team of two), $\binom{4}{2} = 6$.

3.3 Equally likely outcome models

If a finite sample space Ω has N outcomes and each has probability $1/N$, then for any event $A \subseteq \Omega$,

$$P(A) = \frac{|A|}{|\Omega|} = \frac{|A|}{N}.$$

Example (cards). From a standard deck (ignore suits if needed), probability of drawing a specific rank among 13 equally likely ranks is $1/13$.

Example (bitstrings). Choose a length-n bitstring uniformly from $\{0, 1\}^n$. The number with exactly k ones is $\binom{n}{k}$, so

$$P(\text{exactly } k \text{ ones}) = \frac{\binom{n}{k}}{2^n}.$$

3.4 Sanity checks using counting trees

Trees are useful as quick structural checks that the product rule and $\binom{n}{k}$ counts match the process.

3.5 Worked examples

Example (teams from a class). From n students, how many distinct teams of k can be formed? Order does not matter:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

If roles are distinct (e.g., captain and co-captain), then order matters:

$$P(n, 2) = n(n-1).$$

Example (quality control). A batch has n items; d defective, $n-d$ good. If two are chosen at random without replacement, the probability that both are good is

$$\frac{\binom{n-d}{2}}{\binom{n}{2}}.$$

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Curiosity

Where might an "equally likely" assumption fail? Propose one variation of a counting model that invalidates symmetry.

Connections

Connect counting to later topics: binomial and negative binomial models, Poisson approximations, and reliability compositions that rely on combinatorial structure.

Creating Value

Design a simple sampling or inspection plan for a real decision (e.g., quality control or scheduling), and explain how counting supports a defensible choice.

Practice

A. Remember and Understand

1. [Remember]State the product rule and give one short example.
2. [Understand]Explain the difference between $P(n,k)$ and $\binom{n}{k}$ with a concrete scenario.
3. [Understand]In the equally likely model on a finite Ω, explain why $P(A) = |A|/|\Omega|$.

B. Apply and Analyze

1. [Apply]A code has 3 letters (A, B, C) followed by 2 digits (0–9). Count how many codes if letters can repeat and digits can repeat.
2. [Apply]From 10 students, how many ordered pairs (captain, co-captain) can be formed? How many unordered pairs?
3. [Analyze]A length-8 bitstring is chosen uniformly. Compute $P(\text{exactly 3 ones})$ and $P(\text{at least 3 ones})$.

C. Evaluate and Create

1. [Evaluate]In a simple hiring process with two rounds (screen, interview), propose a counting model that matches capacity constraints and justify it.
2. [Create]Create a three-step tree for a process of your choice and use it to compute a count both from the tree and from a closed-form expression; explain the match.