Chapter 2

Events, Axioms, and Probability Models

Topics to Cover

- Sample spaces and events
- Event operations and diagrams
- Probability models and Kolmogorov's axioms
- Consequences: complement rule, addition rule, inclusion–exclusion, union bound
- σ-algebras as collections of allowable events

Learning Objectives

- Describe a sample space and event structure for simple systems.
- State the three probability axioms in plain language and formal notation.
- Manipulate events using complements, unions, intersections, and disjointness.
- Derive basic identities such as $P(A^c) = 1 - P(A)$ and $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
- Explain what a σ-algebra is and why probability models are built on them.

2.1 Sample spaces and events

Before assigning probabilities, we must specify what can happen. A **sample space** Ω is the set of all possible outcomes. Examples:

- Rolling a die: $\Omega = \{1, 2, 3, 4, 5, 6\}$.
- Weather tomorrow: $\Omega = \{\text{rain}, \text{no rain}\}$.
- Packet arrival in a time slot: $\Omega = \{0, 1\}$ (no arrival or arrival).

An **event** is a subset of Ω. For the die example:

$$A = \{2, 4, 6\} \qquad \text{('even number')}.$$

Events are the objects to which we will assign probabilities.

2.2 Event operations

Given events A and B:

- A^c: complement—outcomes where A does not occur.
- $A \cup B$: union—outcomes where at least one occurs.
- $A \cap B$: intersection—outcomes where both occur.
- A and B are **disjoint** if $A \cap B = \emptyset$.

Venn diagrams

The following simple TikZ diagrams illustrate these operations.

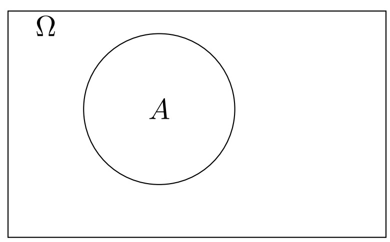

Figure 2.1: Event A inside sample space Ω.

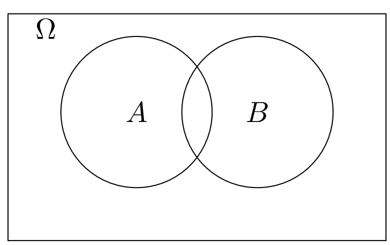

Figure 2.2: Events A and B and their overlap.

2.3 Probability models and axioms

A **probability model** consists of:

- a sample space Ω,
- a collection of allowable events $\mathcal{F}$,
- a probability assignment $P(\cdot)$.

2.3.1 The need for a collection of events

Not every subset of Ω needs to be (or should be) an event. In simple systems $\mathcal{F}$ includes all subsets, but in richer systems we only allow events we can describe or reason about.

A **σ-algebra** $\mathcal{F}$ is a collection of subsets of Ω satisfying:

1. $\Omega \in \mathcal{F}$.
2. If $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$.
3. If $A_1, A_2, \dots \in \mathcal{F}$ then $\bigcup_{n\geq 1} A_n \in \mathcal{F}$.

This guarantees that the usual event operations stay within the model.

2.3.2 Kolmogorov's axioms

A probability measure $P : \mathcal{F} \to [0, 1]$ satisfies:

1. $P(A) \geq 0$ for every $A \in \mathcal{F}$.
2. $P(\Omega) = 1$.
3. For disjoint $A_1, A_2, \dots,$

$$P\left(\bigcup_{n\geq 1} A_n\right) = \sum_{n\geq 1} P(A_n).$$

2.4 Consequences of the axioms

Complement rule.

$$P(A^c) = 1 - P(A).$$

Monotonicity. If $A \subseteq B$ then $P(A) \leq P(B)$.

Addition rule (two events).

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Union bound.

$$P(A \cup B) \leq P(A) + P(B).$$

This will reappear in reliability calculations.

2.5 Examples

2.5.1 Two sensors

A system has two sensors that can fail. Let:

$$A = \{\text{sensor 1 fails}\}, \qquad B = \{\text{sensor 2 fails}\}.$$

If $P(A) = 0.1$, $P(B) = 0.1$, and they fail independently, then

$$P(A \cup B) = 0.1 + 0.1 - 0.1 \cdot 0.1 = 0.19.$$

If independence is not known:

$$P(A \cup B) \leq 0.1 + 0.1 = 0.2.$$

2.5.2 Weather

Let:

$$A = \{\text{rain}\}, \qquad A^c = \{\text{no rain}\}.$$

Then $P(A^c) = 1 - P(A)$.

When the forecaster reports a number p, the model in Chapter 1 describes how payoffs depend on events A and A^c.

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Curiosity

What hidden assumptions lie behind defining a sample space? How might different teams model the same situation differently?

Connections

Event operations introduced here connect directly to the Law of Total Probability, Bayes theorem, and reliability analysis.

Creating Value

Define events clearly in an engineering scenario (sensing, networking, forecasting) so that decisions based on them reliably create value.

Practice

A. Remember and Understand

1. [Remember]Define sample space, event, and σ-algebra in your own words.
2. [Understand]Use the axioms to derive $P(A^c) = 1 - P(A)$.
3. [Understand]Draw a Venn diagram showing $A \cup B$, $A \cap B$, and A^c.

B. Apply and Analyze

1. [Apply]Compute $P(A \cup B)$ when $P(A) = 0.3$, $P(B) = 0.4$, and $P(A \cap B) = 0.1$.
2. [Analyze]A system fails if either component 1 or 2 fails. If $P(\text{fail}_1) = 0.1$, $P(\text{fail}_2) = 0.1$, give upper and lower bounds on system failure.
3. [Analyze]Let $A \subseteq B$. Show directly from the axioms that $P(A) \leq P(B)$.

C. Evaluate and Create

1. [Evaluate]For a sensing system, propose a sample space and three events relevant to an operations team.

2. [Create]Formulate a σ-algebra of meaningful events for a simple weather-based decision (e.g., cancel, delay, proceed).