Chapter 1

Course Introduction and Embracing Uncertainty

Topics to Cover

- What probability means in engineering applications
- Expected value and risk–neutral decisions
- Forecasting and incentives through payoff-based interpretations

Learning Objectives

- Identify actions, outcomes, probabilities, and payoffs in simple decision problems.
- Compute expected value and apply a risk–neutral decision rule.
- Derive and interpret why a payoff rule leads a forecaster to report her true belief.

1.1 Why probability

If a fair die is rolled, we say the probability of rolling a six is 1/6. How do we know? If the die is rolled a million times, about one-sixth of the outcomes are sixes.

If a weather forecaster says "the probability of rain tomorrow is 30%," how should we interpret that? What does "30%" mean? One way to understand probability is to look at the *incentives* under which a forecaster makes a report. If we understand the rules determining her payoffs, then we gain structural clarity about what her reported number represents.

This chapter introduces two examples that motivate these ideas.

1.2 Example A: stock decision (risk–neutral baseline)

Tomorrow's stock movement falls into one of:

$$\text{Very High (V)}, \quad \text{High (H)}, \quad \text{Low (L)}.$$

Let returns be r_V, r_H, r_L, and subjective probabilities be p_V, p_H, p_L.

A risk–neutral decision maker buys the stock if the expected return is positive:

$$\mathbb{E}[R] = p_V r_V + p_H r_H + p_L r_L > 0.$$

Figure 1.1 summarizes the structure.

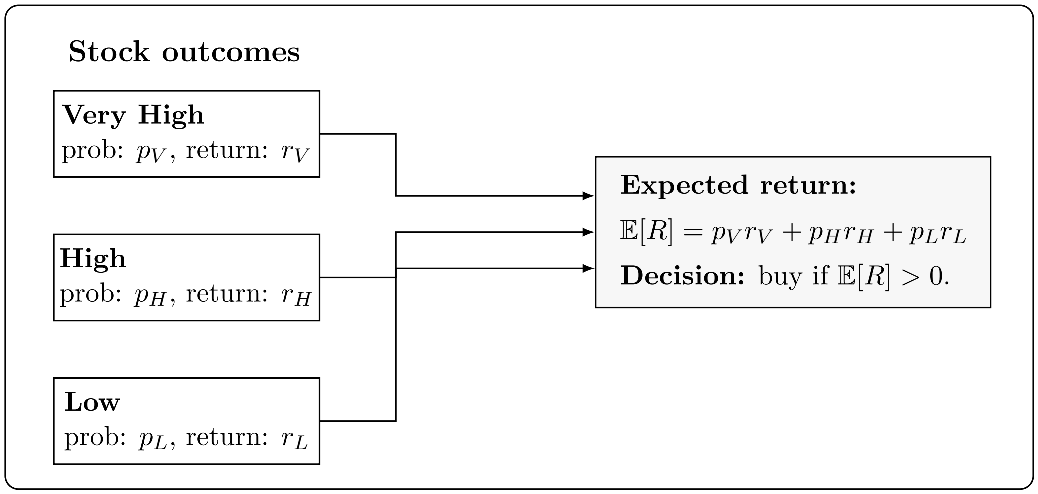

Figure 1.1: Three–tier model for tomorrow's price and the risk–neutral decision rule.

1.3 Example B: rain forecasting and incentives

Suppose a weather forecaster reports a number p (between 0 and 1) as her declared 'probability of rain tomorrow.' Let q be her *true* belief about the chance of rain.

The payment rule is as follows. If it rains tomorrow, she receives p. Otherwise, she keeps the initial p, but must pay $-\ln(1-p)$. So her payoff is:

- Rain: p
- No rain: $p + \ln(1-p)$

For an arbitrary true chance of rain $q \in (0,1)$, if the forecaster reports $p \in (0,1)$, her expected payoff is:

$$F(p) = q\,p + (1-q)\big(p + \ln(1-p)\big) = p + (1-q)\ln(1-p).$$

The forecaster chooses the report p that maximizes this quantity:

$$\hat{p} \;=\; \arg\max_{0<p<1} \; F(p).$$

Differentiate with respect to p:

$$F'(p) \;=\; 1 - \frac{1-q}{1-p}.$$

Setting $F'(p) = 0$ gives

$$1 - \frac{1-q}{1-p} = 0 \quad\Longleftrightarrow\quad 1-p = 1-q \quad\Longleftrightarrow\quad \hat{p} = q.$$

Since

$$F''(p) = -\frac{1-q}{(1-p)^2} < 0 \qquad (0 < p < 1),$$

the function $F(p)$ is strictly concave and $\hat{p} = q$ is the unique maximizer.

In words, for any fixed belief q, the forecaster maximizes her expected payoff by reporting $p = q$. Figure 1.2 summarizes the structure.

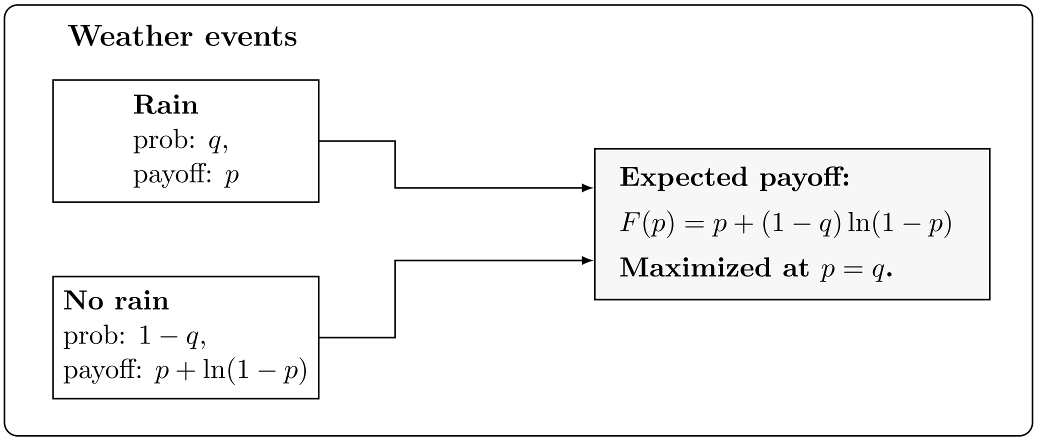

Figure 1.2: Rain forecasting diagram under the payoff rule rain $\to p$ and no rain $\to p + \ln(1 - p)$.

1.4 Modeling checklist

1. Outcomes and events: identify what can happen.
2. Probabilities: quantify beliefs or frequencies for these outcomes.
3. Actions and payoffs: list the decisions and their consequences.
4. Decision rule: choose the action maximizing expected value.

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Curiosity

Explore how decisions change if outcome categories are refined or merged. Identify what additional information reduces uncertainty.

Connections

Connect expected value reasoning here to the Law of Total Probability, Bayes theorem, and reliability approximations using bounds introduced later.

Creating Value

Propose a payoff or scoring rule that encourages honest weather forecasts for campus operations planning outdoor events.

Practice

A. Remember and Understand

1. [Remember]List actions, outcomes, probabilities, and returns in the stock example.
2. [Understand]State the payoff under the forecasting rule if it rains and if it does not rain.
3. [Understand]Compute $\mathbb{E}[R]$ in the stock example with $(3, 1, -1)$ and $(0.2, 0.5, 0.3)$.

B. Apply and Analyze

1. [Apply]Compute $\mathbb{E}[R]$ for $(4, 1, -2)$ and $(0.2, 0.6, 0.2)$ and decide.
2. [Apply]Compute the rain payoff p and no-rain payoff $p + \ln(1 - p)$ for $q = 0.3$ and $p = 0.4$.
3. [Analyze]For $F(p) = p + (1 - q)\ln(1 - p)$ compute $F'(p)$ and verify $F'(q) = 0$.

C. Evaluate and Create

1. [Evaluate]Explain why reporting p very close to 0 or 1 may be risky under this rule if q is moderate.
2. [Create]Propose a threshold-based decision rule (e.g., cancel an event if p exceeds a threshold) and justify it.